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On handlebody-links and Milnor's link-homotopy invariants

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1 Introduction

This is a survey of the joint work [13] with Atsuhiko Mizusawa.

A *handlebody-link* [11, 27] is a disjoint union of embeddings of handlebodies in the 3-sphere S^3 (Figure 1). Two handlebody-links are *equivalent* if there is an ambient iso-

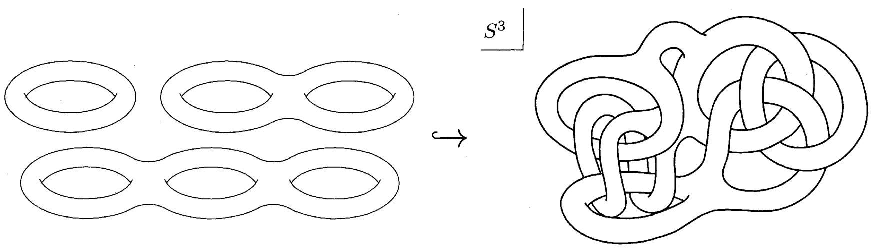

FIGURE 1. A handlebody-link.

topy which transforms one to the other. An *HL-homotopy* is an equivalence relation on handlebody-links, which is analogous to link-homotopy of links. Here, *link-homotopy* is generated by ambient isotopies and self-crossing changes. In [22], Mizusawa and Nikkuni showed that the HL-homotopy classes of 2-component handlebody-links were classified completely by the linking numbers for handlebody-links, which was defined by Mizusawa in [21]. In [13], we construct HL-homotopy invariants for handlebody-links by using Milnor's $\overline{\mu}$-invariants for links. We then give a necessary and sufficient condition of that a handlebody-link is HL-homotopic to a separable one by the extended Milnor's $\overline{\mu}$-invariants. Here, a handlebody-link is *separable* if there exists a disjoint union of 3-balls such that each component of the handlebody-link is contained in a distinct 3-ball. Moreover, we give a bijection between the set of HL-homotopy classes of n-component handlebody-links with some assumption and a quotient of a tensor product of $\mathbb{Z}-$modules by the action of the general linear group.

2 Preliminaries

J. Milnor defined a family of invariants for an ordered oriented link in S^3 as a generalization of the linking numbers, in [19, 20]. These invariants are called *Milnor's $\overline{\mu}$-invariants.* For an ordered oriented n-component link L, Milnor's $\overline{\mu}$-invariant is specified by a sequence I of indices in $\{1, 2, \dots, n\}$ and denoted by $\overline{\mu}_L(I)$. If the sequence is with distinct indices, then this invariant is also link-homotopy invariant and called *Milnor's link-homotopy invariant.*

We introduce the definition of Milnor's link-homotopy invariants, and to give invariants for handlebody-links, we show that these are additive under a bund sum for components.

Let $L = L_1 \cup \cdots \cup L_n$ be an ordered oriented n-component link in S^3. Consider the link group $\pi = \pi_1(S^3 \setminus L_1 \cup \cdots \cup L_{n-1})$ of $L_1 \cup \cdots \cup L_{n-1}$ and denote the i-th meridian by m_i for i $(1 \leq i \leq n-1)$.

Given a finitely generated group G, the *reduced group $\overline{G}$* is defined to the quotient of G by its normal subgroup generated by $[g, hgh^{-1}]$ for any $g, h \in G$, where $[a, b]$ means the commutator of a and b. Then $\overline{\pi}$ is generated by the meridians $m_1, m_2, \dots, m_{n-1}$.

Let $\mathbb{Z}[[X_1, \dots, X_{n-1}]]$ be the non-commutative formal power series ring generated by $X_1, \dots, X_{n-1}$. Denote by $\hat{Z}$ its quotient ring by the two-side ideal generated by all monomials in which at least one of the generators appear at least twice. The *Magnus expansion* φ is a homomorphism from the free group $F(m_1, \dots, m_{n-1})$ generated by $m_1, \dots, m_{n-1}$ into $\mathbb{Z}[[X_1, \dots, X_{n-1}]]$, defined by sending m_i to $1 + X_i$ and m_i^{-1} to $1 - X_i + X_i^2 - \cdots$. It induces a homomorphism from $\overline{F(m_1, \dots, m_{n-1})}$ into $\hat{Z}$. Let $w_n \in F(m_1, \dots, m_{n-1})$ be a word representing L_n in $\overline{\pi}$. We then define $\mu_L(i_1 i_2 \dots i_r n)$ for distinct indices $i_1, i_2, \dots, i_r, n$ as the coefficient of the Magnus expansion of w_n in $\hat{Z}$:

$$\varphi(w_n) = 1 + \sum \mu_L(i_1 i_2 \dots i_r n) X_{i_1} X_{i_2} \dots X_{i_r},$$

where the summation is over all sequences $i_1 i_2 \dots i_r$ with distinct indices between 1 and $n-1$. Similarly, we define $\mu_L(i_1 i_2 \dots i_s)$ for any distinct indices between 1 and n. We define $\overline{\mu}_L(i_1 i_2 \dots i_r n)$ as the residue class of $\mu_L(i_1 i_2 \dots i_r n)$ modulo the indeterminacy $\Delta_L(i_1 i_2 \dots i_r n)$ which is the greatest common divisor of $\mu_L(j_1 j_2 \dots j_s)$'s, where $j_1 j_2 \dots j_s$ ranges over all sequences obtained by deleting at least one of the indices $i_1, i_2, \dots, i_r, n$ and permuting the remaining ones cyclicly. Moreover we define $\Delta_L(i_1 n) = 0$. Similar to this, for any n-component link L, we can define $\overline{\mu}_L(I)$ for any sequence I of distinct indices in $\{1, 2, \dots, n\}$

Theorem 2.1 ([19, 20]). *If L and L' are link-homotopic, then $\overline{\mu}_L(I) = \overline{\mu}_{L'}(I)$ for any sequence I with distinct indices.*

Lemma 2.2 ([20]). *Let L be an ordered oriented link. Then the following relations hold.*
(1) $\overline{\mu}_L(i_1 i_2 \ldots i_m) = \overline{\mu}_L(i_2 \ldots i_m i_1)$
(2) If the orientation of the k-th component of L is reversed, then $\overline{\mu}_L(i_1 i_2 \ldots i_m)$ is multiplied by $+1$ or -1 according as the sequence $i_1 i_2 \ldots i_m$ contains k an even or odd number of times.

The following lemma is used for Proposition 3.4. This lemma is showed by using the definition of Milnor's link-homotopy invariants.

Lemma 2.3. *Let $L = L_1 \cup L_2 \cup \cdots \cup L_{n-1}$ be an $(n-1)$-component link in S^3. Let K and K' be disjoint knots in $S^3 \setminus L$. Let I be a sequence with distinct indices in $\{1, 2, \ldots, n\}$. If I contains the index n,*

$$\mu_{L \cup (K \sharp_b K')}(I) \equiv \mu_{L \cup K}(I) + \mu_{L \cup K'}(I) \mod \gcd(\Delta_{L \cup K}(I), \Delta_{L \cup K'}(I)),$$

where $K \sharp_b K'$ is a band sum of K and K' with respect to any band, and $L \cup (K \sharp_b K')$, $L \cup K$ and $L \cup K'$ are n-component links whose n-th components are $K \sharp_b K'$, K and K', respectively.

Remark 2.4. By a property of the $\bar{\mu}$-invariant, we can obtain the same result for a band sum of the i-th component instead of the n-th component.

Remark 2.5. In [14], V. S. Krushkal showed Milnor's $\overline{\mu}$-invariants are additive under connected sum for links which are separated by a 2-sphere.

3 Milnor's $\overline{\mu}$-invarinats for handlebody-links

In this section, we define the HL-homotopy, which is an equivalence relation on handlebody-links and construct HL-homotopy invariants for handlebody-links by using Milnor's $\overline{\mu}$-invariants.

Definition 3.1 (HL-homotopy). Let H_0 be n handlebodies and H_i $(i = 1, 2)$ two n-component handlebody-links obtained by embedding H_0 to S^3 by f_i. Two handlebody-links H_1 and H_2 are called *HL-homotopic* if there is homotopy h_t from f_1 to f_2 where the components of $h_t(H_0)$ are mutually disjoint at any $0 \leq t \leq 1$.

Remark 3.2. In [22], the notation of *neighborhood homotopy* of spatial graphs was introduced. A spatial graph is an embedding of graph in S^3. We can represent the HL-homotopy of handlebody-links by the neighborhood homotopy of spatial graphs.

Let $H = L_1 \cup \cdots \cup L_n$ be an n-component handlebody-link with genus g_i for each i. Let $\{e^i_1, \ldots, e^i_{g_i}\}$ be a basis of $H_1(L_i; \mathbb{Z})$ and $\mathcal{B} = \{e^1_1, \ldots, e^1_{g_1}, \ldots, e^n_1, \ldots, e^n_{g_n}\}$. We can regard an element of $\mathcal{B}$ as an embedded closed oriented circle in S^3. So the disjoint union $e^1_{k_1} \cup e^2_{k_2} \cup \cdots \cup e^n_{k_n}$ can be regarded as an ordered oriented link for each k_i $(1 \leq k_i \leq g_i)$. Let I be a sequence of length m $(m \leq n)$ with distinct indices in $\{1, 2, \ldots, n\}$. For each I, we define an element $M_{H,\mathcal{B}}(I)$ of tensor product space $(\mathbb{Z}/\Delta_I\mathbb{Z})^{g_1} \otimes \cdots \otimes (\mathbb{Z}/\Delta_I\mathbb{Z})^{g_n}$ as $\mathbb{Z}/\Delta_I\mathbb{Z}$-module defined by

$$M_{H,\mathcal{B}}(I) := \sum_{k_1,\ldots,k_n=1}^{g_1,\ldots,g_n} \overline{\mu}_{e^1_{k_1} \cup \cdots \cup e^n_{k_n}}(I)\ \boldsymbol{e}^1_{k_1} \otimes \cdots \otimes \boldsymbol{e}^n_{k_n},$$

where $\overline{\mu}_{e^1_{k_1} \cup \cdots \cup e^n_{k_n}}(I)$ is in $\mathbb{Z}/\Delta_I\mathbb{Z}$, Δ_I is the greatest common divisor of all $\Delta_{e^1_{k_1} \cup \cdots \cup e^n_{k_n}}(I)$ for all $k_1, \ldots, k_n$, where $\Delta_{e^1_{k_1} \cup \cdots \cup e^n_{k_n}}(I)$ is indeterminacy of the original Milnor's invariant for the link $e^1_{k_1} \cup e^2_{k_2} \cup \cdots \cup e^n_{k_n}$ and $\boldsymbol{e}^i_{k_i}$ is the canonical basis $(0, \ldots, 0, \overset{k_i}{\check{1}}, 0, \ldots, 0)$ of $(\mathbb{Z}/\Delta_I\mathbb{Z})^{g_i}$ as $\mathbb{Z}/\Delta_I\mathbb{Z}$-module.

Remark 3.3. If the first homology group of each component of H is $\mathbb{Z}$, the $M_{H,\mathcal{B}}(I)$ is identified with the original Milnor's link-homotopy invariant for a link, essentially.

We consider a natural action of $GL(g_1, \mathbb{Z}) \times \cdots \times GL(g_n, \mathbb{Z})$ on $(\mathbb{Z}/\Delta_I\mathbb{Z})^{g_1} \otimes \cdots \otimes (\mathbb{Z}/\Delta_I\mathbb{Z})^{g_n}$ and denote by $M_H(I)$ the residue class of $M_{H,\mathcal{B}}(I)$ by the action for $(\mathbb{Z}/\Delta_I\mathbb{Z})^{g_1} \otimes \cdots \otimes (\mathbb{Z}/\Delta_I\mathbb{Z})^{g_n}$.

Proposition 3.4. *Let H be an n-component handlebody-link. Then $M_H(I)$ is independent of a basis $\mathcal{B}$ of $H_1(H, \mathbb{Z})$ and an HL-homotopy invariant.*

Proof. The proof is by induction on the length m of sequence I. We can show it by using properties of $\overline{\mu}$-invariants for links (Lemma 2.2 and 2.3). See [13] for details. □

Example 3.5. Let H be a handlebody-link which are the regular neighborhood of graph illustrated in Figure 2. Let $I = 123$. Then, $\Delta_{e^1_1 \cup e^2_1 \cup e^3_1}(I) = \Delta_{e^1_1 \cup e^2_1 \cup e^3_2}(I) = 2$ and $\Delta_{e^1_{k_1} \cup e^2_{k_2} \cup e^3_{k_3}}(I) = 0$ in other cases. So $\Delta_I = 2$ and

$$M_H(I) = 1\ \boldsymbol{e}^1_1 \otimes \boldsymbol{e}^2_1 \otimes \boldsymbol{e}^3_1 + 1\ \boldsymbol{e}^1_2 \otimes \boldsymbol{e}^2_2 \otimes \boldsymbol{e}^3_2 \in (\mathbb{Z}_2)^2 \otimes (\mathbb{Z}_2)^2 \otimes (\mathbb{Z}_2)^2.$$

We can show the following corollary by using clasper theory introduced by Habiro [8].

Corollary 3.6. *An n-component handlebody-link H is HL-homotopic to a separable handlebody-link if and only if $M_H(I) = 0$ for any I.*

Remark 3.7. T. Fleming defined a numerical invariant $\lambda_\Phi(H)$ of a pair of a spatial graph Φ and its subgraph H under component homotopy in [3]. Now, we define Φ as a handlebody-link instead of a spatial graph and H as its component instead of a subgraph. We then can naturally extend this invariant to a pair of a handlebody-link and its component under HL-homotopy. Then, the value of $\lambda_\Phi(H)$ is the length of first non-vanishing for $M_\Phi(I)$ such that I contains the component number of H.

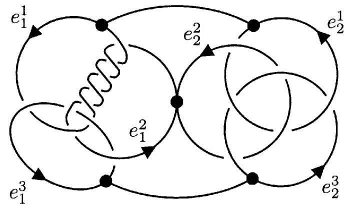

FIGURE 2. Handlebody-link H.

4 Main Theorem

Let $\mathbb{H}[g_1, g_2, \cdots, g_n]$ be the set of n-component handlebody-links with genus g_i for each $1 \leq i \leq n$ such that its any $(n-1)$-component subhandlebody-link is HL-homotopic to a separable handlebody-link. By Corollary 3.6, this condition is equivalent to that its any $M(I)$'s of length less than n vanishes.

Let S be a permutation group on $\{2, 3, \ldots, n-1\}$. For any element σ in S, we define I_σ as a sequence $1\sigma(23\cdots n-1)n$.

Theorem 4.1. *For any element σ in S, the map*

$$\begin{aligned} \varphi : \mathbb{H}[g_1, \cdots, g_n] &\to \oplus_{\sigma\in S}(\mathbb{Z}^{g_1} \otimes \cdots \otimes \mathbb{Z}^{g_n}) \\ H &\mapsto (M_H(I_\sigma))_{\sigma\in S} \end{aligned}$$

induces a bijection between the set of HL-homotopy classes of $\mathbb{H}[g_1, g_2, \cdots, g_n]$ and the residue class of $\oplus_{\sigma\in S}(\mathbb{Z}^{g_1} \otimes \cdots \otimes \mathbb{Z}^{g_n})$ by diagonal action of general linear group.

We give two examples.

Example 4.2. Let $I = 123$. Let H_1 and H_2 be two handlebody-links which are the regular neighborhood of graphs depicted in Figure 3. Then, $\Delta_I = 0$ and

$$\begin{aligned} M_{H_1}(I) =& 1\ \boldsymbol{e}_1^1 \otimes \boldsymbol{e}_1^2 \otimes \boldsymbol{e}_1^3 + 1\ \boldsymbol{e}_1^1 \otimes \boldsymbol{e}_2^2 \otimes \boldsymbol{e}_1^3 + 1\ \boldsymbol{e}_1^1 \otimes \boldsymbol{e}_3^2 \otimes \boldsymbol{e}_1^3 \\ &+ 2\ \boldsymbol{e}_1^1 \otimes \boldsymbol{e}_1^2 \otimes \boldsymbol{e}_2^3 + 2\ \boldsymbol{e}_1^1 \otimes \boldsymbol{e}_2^2 \otimes \boldsymbol{e}_2^3 + 2\ \boldsymbol{e}_1^1 \otimes \boldsymbol{e}_3^2 \otimes \boldsymbol{e}_2^3 \\ &\in \mathbb{Z}^2 \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^2. \\ M_{H_2}(I) =& 1\ \boldsymbol{e}_1^1 \otimes \boldsymbol{e}_1^2 \otimes \boldsymbol{e}_1^3 + 1\ \boldsymbol{e}_1^1 \otimes \boldsymbol{e}_2^2 \otimes \boldsymbol{e}_1^3 + 1\ \boldsymbol{e}_2^1 \otimes \boldsymbol{e}_1^2 \otimes \boldsymbol{e}_1^3 + 1\ \boldsymbol{e}_2^1 \otimes \boldsymbol{e}_2^2 \otimes \boldsymbol{e}_1^3 \\ & 1\ \boldsymbol{e}_1^1 \otimes \boldsymbol{e}_1^2 \otimes \boldsymbol{e}_2^3 + 1\ \boldsymbol{e}_1^1 \otimes \boldsymbol{e}_2^2 \otimes \boldsymbol{e}_2^3 + 1\ \boldsymbol{e}_2^1 \otimes \boldsymbol{e}_1^2 \otimes \boldsymbol{e}_2^3 + 1\ \boldsymbol{e}_2^1 \otimes \boldsymbol{e}_2^2 \otimes \boldsymbol{e}_2^3 \\ &\in \mathbb{Z}^2 \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^2. \end{aligned}$$

We have that $M_{H_1}(I)$ is transformed to $M_{H_2}(I)$ by the diagonal action of general linear group. Therefore H_1 and H_2 are HL-homotopic.

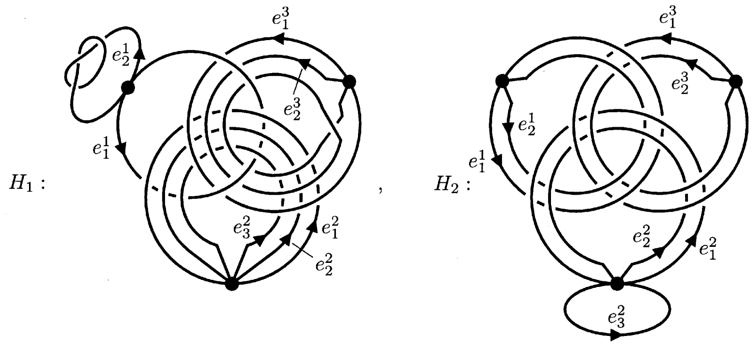

FIGURE 3. Handlebody-links H_1 and H_2.

Example 4.3. Let $I = 123$. Let H_3 and H_4 be two handlebody-links which are the regular neighborhood of graphs depicted in Figure 4. Then, $\Delta_I = 0$ and

$$\begin{aligned} M_{H_3}(I) =& 1\ \boldsymbol{e}^1_1 \otimes \boldsymbol{e}^2_1 \otimes \boldsymbol{e}^3_1 + 1\ \boldsymbol{e}^1_1 \otimes \boldsymbol{e}^2_2 \otimes \boldsymbol{e}^3_1 + 1\ \boldsymbol{e}^1_2 \otimes \boldsymbol{e}^2_1 \otimes \boldsymbol{e}^3_1 \\ & + 1\ \boldsymbol{e}^1_2 \otimes \boldsymbol{e}^2_2 \otimes \boldsymbol{e}^3_1 + 1\ \boldsymbol{e}^1_1 \otimes \boldsymbol{e}^2_3 \otimes \boldsymbol{e}^3_2 + 1\ \boldsymbol{e}^1_2 \otimes \boldsymbol{e}^2_3 \otimes \boldsymbol{e}^3_2 \\ & \in \mathbb{Z}^2 \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^2. \\ M_{H_4}(I) =& 2\ \boldsymbol{e}^1_1 \otimes \boldsymbol{e}^2_1 \otimes \boldsymbol{e}^3_1 + 2\ \boldsymbol{e}^1_2 \otimes \boldsymbol{e}^2_1 \otimes \boldsymbol{e}^3_1 + 1\ \boldsymbol{e}^1_1 \otimes \boldsymbol{e}^2_2 \otimes \boldsymbol{e}^3_2 + 1\ \boldsymbol{e}^1_2 \otimes \boldsymbol{e}^2_2 \otimes \boldsymbol{e}^3_2 \\ & \in \mathbb{Z}^2 \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^2. \end{aligned}$$

We can show that H_1 is not HL-homotopic to H_2 by using some invariants for the action of general linear group on the tensor product space. See [13] for details.

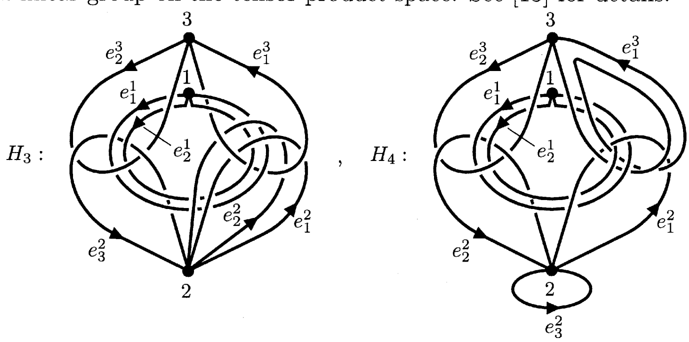

FIGURE 4. Handlebody-links H_3 and H_4.

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